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Properties of the Zeros of Confluent Hypergeometric Functions*

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Several infinite systems of nonlinear algebraic equations satisfied by the zeros of confluent hypergeometric functions are derived. Certain sum rules and other related properties for the zeros follow from these equations. A large class of special functions, which are special cases of confluent hypergeometric functions, is included. This is illustrated in the case of the zeros of Bessel functions and Laguerre polynomials.

1. INTRODUCTION

Recently a series of investigations have been carried out on the properties of the zeros of special functions [2–8, 11, 12, 16, 17]. The functions mainly dealt with were the classical orthogonal polynomials (Jacobi, Laguerre and Hermite) and the Bessel functions. The motivation of this paper is to formulate a unified theory describing these properties. This is initiated by investigating the properties of the zeros of confluent hypergeometric functions—which include a large class of functions as special cases. Indeed the known results concerning the zeros of Bessel functions and Laguerre polynomials [3–5, 7] are seen to follow easily.

Though some properties of the zeros of confluent hypergeometric functions were already known for quite some time [9, 10, 13, 18, 19], we shall mainly concentrate in deriving a new class of infinite systems of nonlinear algebraic equations satisfied by the zeros of the confluent hypergeometric function $\Phi(a, c; x)$ [also denoted by ${}_1F_1(a, c; x)$] [13]. As will be evident later, these equations yield in a natural and straightforward manner some remarkable properties for the zeros of $\Phi(a, c; x)$ and related functions.

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2. Notations and Preliminaries

Let x_j be the infinite (real and complex) zeros of the confluent hypergeometric function $\Phi(a, c; x)$

$$\Phi(a, c; x_j) = 0, \tag{2.1}$$

the infinite series representation for $\Phi(a, c; x)$ being

$$\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!}, \tag{2.2}$$

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1); \qquad (a)_0 = 1. \tag{2.3}$$

The zeros x_j are functions of the parameters a and c, where $c \neq -m$, $m = 0, 1, 2,...$, but for notational convenience we shall not indicate this explicitly.

The infinite product representation for $\Phi(a, c; x)$ is [14, 18]

$$\Phi(a, c; x) = \exp(ax/c) \prod_{j=1}^{\infty} [1 - x/x_j) \exp(x/x_j)]. \tag{2.4}$$

However, throughout this paper we shall deal with the analytic function

$$\chi(a, c; x) = \exp(-ax/c)\, \Phi(a, c; x) = \prod_{j=1}^{\infty} [(1 - x/x_j) \exp(x/x_j)] \tag{2.5}$$

whose zeros of course coincide with the zeros of $\Phi(a, c; x)$ except possibly at infinity.

Buchholz [9, 10] ordered the zeros x_j in such a way that $|x_1| \leqslant |x_2| \leqslant |x_3| \leqslant ...$, and derived formulas for

$$S_p = \sum_{j=1}^{\infty} x_j^{-p}, \qquad p = 2, 3,.... \tag{2.6}$$

For instance, for $p = 2, 3, 4, 5$, these are

$$S_2 = \sum_{j=1}^{\infty} x_j^{-2} = \frac{a(a-c)}{c^2(c+1)}, \tag{2.7}$$

$$S_3 = \sum_{j=1}^{\infty} x_j^{-3} = \frac{a(a-c)(c-2a)}{c^3(c+1)(c+2)}, \tag{2.8}$$

$$S_4 = \sum_{j=1}^{\infty} x_j^{-4} = \frac{a(a-c)[a(a-c)(5c+6) + c^2(c+1)]}{c^4(c+1)^2(c+2)(c+3)}, \tag{2.9}$$

$$S_5 = \sum_{j=1}^{\infty} x_j^{-5} = \frac{a(a-c)(c-2a)[a(a-c)(7c+12) + c^2(c+1)]}{c^5(c+1)^2(c+2)(c+3)(c+4)}. \tag{2.10}$$

3. An Infinite System of Nonlinear Equations

Since the confluent hypergeometric function $\Phi(a, c; x)$ satisfies the differential equation [13]

$$x\Phi'' + (c - x)\,\Phi' - a\Phi = 0 \tag{3.1}$$

it follows from (2.5) that the function $\chi(a, c; x)$ satisfies the differential equation

$$c^2 x\chi'' + c[(2a - c)\,x + c^2]\,\chi' + a(a - c)\,x\chi = 0. \tag{3.2}$$

THEOREM 1. *The zeros x_j of $\chi(a, c; x)$ satisfy the infinite system of nonlinear algebraic equations*

$$2cx_j^2 \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} = (c - 2a)\,x_j - c(c + 2). \tag{3.3}$$

Proof. Differentiating (2.5) logarithmically gives

$$\chi'/\chi = x \sum_{j=1}^{\infty} x_j^{-1}(x - x_j)^{-1}. \tag{3.4}$$

Differentiating again it follows after some trivial algebra

$$\chi''/\chi = \sum_{j=1}^{\infty} x_j^{-2} + 2 \sum_{j=1}^{\infty} x_j^{-1}(x - x_j)^{-1} + 2x^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' [x_j x_k (x - x_j)(x_j - x_k)]^{-1}. \tag{3.5}$$

The last term on the right-hand side follows by interchanging the indices j and k. Further manipulations lead to

$$\chi''/\chi = \sum_{j=1}^{\infty} x_j^{-2} \left[1 + 2(x + x_j)\,x_j \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1}\right] + 2 \sum_{j=1}^{\infty} x_j^{-1}(x - x_j)^{-1} \left[1 + x_j^2 \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1}\right]. \tag{3.6}$$

In the above expressions, as well as the following ones, a prime appended to a sum indicates the exclusion of the singular term. The convergence of the sums appearing in these expressions can be verified a posteriori by the results to follow.

Substituting (3.4) and (3.6) in the differential equation (3.2) we obtain an equation of the form

$$\left[Ax + B + \sum_{j=1}^{\infty} (x - x_j)^{-1} C_j\right] \chi = 0, \tag{3.7}$$

where

$$C_j = 2c^2 x_j \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} - c(c-2a) + \frac{c^2(c+2)}{x_j}, \tag{3.8}$$

$$B = \sum_{j=1}^{\infty} x_j^{-2} \left[c^2 + 2c^2 x_j^2 \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} + c(2a-c)\, x_j\right] + a(a-c) \tag{3.9}$$

and

$$A = 2c^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' [x_j x_k (x_j - x_k)]^{-1}. \tag{3.10}$$

Equation (3.7) immediately implies that C_j, B and A must vanish.

Remark. The vanishing of B and A is consistent with Eq. (3.3) and the convergent sums (2.7) and (2.8). This is easily seen by substituting (2.7) and (3.3) in (3.9); and dividing (3.3) by x_j^3 and using (2.7) and (2.8). Note moreover that $A = 0$ implies that the double sum

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' [x_j x_k (x_j - x_k)]^{-1} \tag{3.11}$$

is antisymmetric with respect to the exchange of the dummy indices j and k, a fact otherwise difficult to establish for such an infinite sum.

4. Higher-Order Nonlinear Equations

The derivation of higher-order nonlinear algebraic equations of the type

$$\sigma_j^{(m)} = x_j^m \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-m} \qquad \text{for} \quad m = 2, 3, 4, 5 \tag{4.1}$$

are presented here. However, the results can be generalized for any m [21].

Theorem 2. *The zeros x_j of the confluent hypergeometric function $\Phi(a, c; x)$ satisfy the relation*

$$2(x_j - x_i)^{-2} = c/2x_j x_i + \sum_{k=1}^{\infty}{}' [(x_j - x_k)(x_i - x_k)]^{-1}. \tag{4.2}$$

Proof. Consider the equation with j replaced by i in (3.3)

$$2cx_i^2 \sum_{t=1}^{\infty}{}' x_t^{-1}(x_i - x_t)^{-1} = (c-2a)\,x_i - 2(c+2); \tag{4.3}$$

multiply (3.3) and (4.3) by x_i and x_j, respectively, subtract (4.3) from (3.3), extract from the sums the terms with $t=j$ and $k=i$ and divide throughout by $(x_j - x_i)$.

THEOREM 3. *The zeros x_j satisfy the nonlinear equations*

$$12x_j^2 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-2} = -x_j^2 + 2(c-2a)\,x_j - c(c+4). \tag{4.4}$$

Proof. Divide (4.2) by x_i, sum over i, omitting of course the singular term with $i=j$, and manipulate the sums algebraically obtaining

$$3 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-2} = (c+1)\left(\sum_{j=1}^{\infty} x_j^{-2} - x_j^{-2}\right) \tag{4.5}$$

$$- \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1}\left[2 + x_j^2 \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1}\right].$$

Now use (2.7) and (3.3).

COROLLARY 3.1. The zeros x_j satisfy the double sum

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' x_j^{-2}(x_j - x_k)^{-2} = \frac{a(a+1)(a+c)(a-c-1)}{4c^2(c+1)^2(c+3)}. \tag{4.6}$$

Proof. Divide (4.4) by x_j^4, sum over j and use (2.7), (2.8) and (2.9).

COROLLARY 3.2. *For real a and c the zeros x_j (which are finite and real) satisfy the bounds*

$$(c-2a) - 2\sqrt{a(a-c)-c} < x_j < (c-2a) + 2\sqrt{a(a-c)-c}. \tag{4.7}$$

Proof. The positivity of the left-hand side of (4.4) implies the positivity of the right-hand side.

THEOREM 4. *The zeros x_j satisfy the nonlinear equations*

$$8x_j^3 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-3} = (c-2a)\,x_j - c(c+2). \tag{4.8}$$

Proof. Divide (4.2) by $(x_j - x_i)$ and sum over i $(i \neq j)$. The double sum appearing in the right-hand side vanishes by antisymmetry. Then use (3.3).

COROLLARY 4.1$_4$ *The double sum*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' (x_k - x_k)^{-3} = 0. \tag{4.9}$$

Proof. Divide (4.8) by x_j^3, sum over j and use (2.7) and (2.8).

COROLLARY 4.2. *For real a and c the smallest (real) zero x_1 satisfies the upper bound*

$$x_1 < \frac{c(c+2)}{c-2a}. \tag{4.10}$$

Proof. The left-hand side of (4.8) is negative for $j = 1$, x_1 being the smallest (real) zero.

Remark. Equations (3.3) and (4.8) imply the identity

$$c \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} = 4x_j \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-3}. \tag{4.11}$$

THEOREM 5. *The zeros x_j satisfy the nonlinear equations*

$$\begin{aligned} 720x_j^4 &\sum_{k=1}^{\infty}{}' (x_j - x_k)^{-4} \\ &= x_j^4 - 4(c-2a)\,x_j^3 + 2[8a(a-c) + c(3c-2)]\,x_j^2 \\ &\quad - 4(c-2a)(c^2 - 2c - 18)\,x_j + c[c(c+4)(c-8) - 72(c+2)]. \end{aligned} \tag{4.12}$$

Proof. Divide (4.2) by $(x_j - x_i)^2$, sum over i $(i \neq j)$ and manipulate algebraically the sums appearing in the right-hand side, getting

$$\begin{aligned} 5x_j^4 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-4} &= cx_j^2 \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} \\ &\quad + x_j^2 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-2} \left[c + x_j^2 \sum_{k=1}^{\infty}{}' (x_j - x_k)^{-2} \right]. \end{aligned} \tag{4.13}$$

Now use (3.3) and (4.4).

THEOREM 6. *The zeros x_j satisfy the nonlinear equations*

$$288x_j^5 \underset{k=1}{\overset{\infty}{\sum}}{}' (x_j - x_k)^{-5}$$
$$= -(c-2a)x_j^3 + [8a(a-c) + c(3c-2)]x_j^2$$
$$- 3(c-2a)(c+2)(c-4)x_j + c(c+4)(c^2 - 8c - 12). \quad (4.14)$$

Proof. Divide (4.2) by $(x_j - x_i)^3$, sum over $i(i \neq j)$, manipulate algebraically the right-hand side and use (3.3), (4.4) and (4.8).

COROLLARY. The double sum

$$\underset{j=1}{\overset{\infty}{\sum}} \underset{k=1}{\overset{\infty}{\sum}}{}' (x_j - x_k)^{-5} = 0. \quad (4.15)$$

Proof. Divide (4.14) by x_j^5 and use (2.7), (2.8), (2.9) and (2.10).

Remark. The double sum

$$\underset{j=1}{\overset{\infty}{\sum}} \underset{k=1}{\overset{\infty}{\sum}}{}' (x_j - x_k)^{-2r-1} = 0, \qquad r = 1, 2, \ldots. \quad (4.16)$$

However, for $r = 0$ this sum does not vanish as can be easily seen by summing (3.3) over j and noting that the reciprocal power sum $\sum_{j=1}^{\infty} x_j^{-1}$ which appears in the result (by partial fraction) is already divergent [9, 10].

Another remarkable property of the nonlinear equations (3.3) is that the sums (2.6) for the reciprocal integral powers of the zeros x_j can be easily derived without resorting to the complex integration technique of Buchholz [9]. For instance, for $p = 4$, dividing (3.3) by x_j^4 and summing over j, we get

$$2c \underset{j=1}{\overset{\infty}{\sum}} \underset{k=1}{\overset{\infty}{\sum}}{}' x_j^{-2} x_k^{-1} (x_j - x_k)^{-1} = = (c-2a) \underset{j=1}{\overset{\infty}{\sum}} x_j^{-3} - c(c+2) \underset{j=1}{\overset{\infty}{\sum}} x_j^{-4}. \quad (4.17)$$

The indices j and k being dummy, the double sum on the left-hand side simplifies to

$$2 \underset{j=1}{\overset{\infty}{\sum}} \underset{k=1}{\overset{\infty}{\sum}}{}' x_j^{-2} x_k^{-1} (x_j - x_k)^{-1} = \underset{j=1}{\overset{\infty}{\sum}} x_j^{-4} - \left(\underset{j=1}{\overset{\infty}{\sum}} x_j^{-2} \right)^2. \quad (4.18)$$

The sum (2.9) then follows using (2.7) and (2.8). Similarly dividing (3.3) by x_j^5 and using (2.7), (2.8) and (2.9), the sum (2.10) is obtained.

5. Zeros of Bessel Functions

The limit formula [1] connecting the confluent hypergeometric function $\Phi(a, c; x)$ to the Bessel function $J_\nu(x)$

$$\lim_{a\to\infty} \Phi(a, \nu+1, -x/a) = \Gamma(\nu+1)\, x^{-\nu/2} J_\nu(2x^{1/2}) \tag{5.1}$$

implies

$$\lim_{a\to\infty} ax_j \to -y_j/4, \tag{5.2}$$

where y_j are the *squares* of the zeros of the Bessel function $J_\nu(x)$. Using this limit there follows from Theorems 1 to 5 respectively the infinite systems of nonlinear equations satisfied by the squares of the zeros of Bessel functions [5, 7]

$$2y_j \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-1} = -(\nu+1), \tag{5.3a}$$

$$12y_j^2 \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-2} = y_j - (\nu+1)(\nu+5), \tag{5.3b}$$

$$16y_j^3 \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-3} = y_j - 2(\nu+1)(\nu+3), \tag{5.3c}$$

$$720y_j^4 \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-4} = y_j^2 - 2(\nu^2 - 19)\, y_j + (\nu+1)(\nu^3 - \nu^2 - 109\nu - 251), \tag{5.3d}$$

$$576y_j^5 \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-5} = y_j^2 - 3(\nu^2 - 9)\, y_j + 2(\nu+1) \times (\nu+5)(\nu^2 - 6\nu - 19), \tag{5.3e}$$

where $\nu \neq -1, -2, -3, \ldots$. In deriving (5.3a) from Theorem 1, the well-known sum rule [11, 20]

$$\sum_{j=1}^{\infty} y_j^{-1} = [4(\nu+1)]^{-1} \tag{5.4}$$

has also been used. Note that Equations (5.3) are more general than those already known [7] as they now hold for both real and complex zeros. This is due to the imposition of fewer restrictions on the values of ν. Note moreover

that when y_j is real, Eqs. (5.3) also provide some interesting bounds. For instance, (5.3b) provides the lower bound

$$y_j > (\nu + 1)(\nu + 5) \tag{5.5}$$

whereas (5.3c) provides for the least zero y_1 the upper bound

$$y_1 < 2(\nu + 1)(\nu + 3). \tag{5.6}$$

Other results concerning the squares of the zeros of Bessel functions of the forms [5, 7]

$$\sum_{j=1}^{\infty} y_j^{-p}, \qquad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty}{}' (y_j - y_k)^{-p}, \qquad p = 2, 3, \ldots,$$

can be easily obtained using the limit (5.1).

Certain results concerning the zeros (not the squares of the zeros) of Bessel functions can also be obtained from the results for the zeros of confluent hypergeometric functions but this time by considering the Bessel function $J_\nu(x)$ as a particular case [1, 13]

$$J_\nu(x) = \frac{1}{\Gamma(\nu + 1)} (x/2)^\nu e^{-ix} \Phi(\nu + \tfrac{1}{2}, 2\nu + 1; 2ix) \tag{5.7}$$

or equivalently the function

$$\phi(x) \equiv \Gamma(\nu + 1)(x/2)^\nu J_\nu(x) = \chi(\nu + \tfrac{1}{2}, 2\nu + 1; 2ix), \tag{5.8}$$

which follows from (2.5). Replacing a by $\nu + \frac{1}{2}$, c by $2\nu + 1$ and x_j by $2ix_j$ in Eq. (3.3) we obtain the nonlinear equations satisfied by the zeros x_j of $\phi(x)$

$$\sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} = -(2\nu + 3)/2x_j^2, \tag{5.9}$$

where $2\nu \neq -1, -2, -3, \ldots$. However, Eq. (5.3a) for the squares of the zeros y_j $(=x_j^2)$ follows from (5.9) by noting that the function $\phi(x)$ given by (5.8) is an even function of x. Indeed the symmetry of the zeros x_j of this function around the origin implies that

$$\begin{aligned} \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} &= \sum_{k=1}^{\infty}{}' x_k^{-1}(x_j - x_k)^{-1} - \sum_{k=1}^{\infty} x_k^{-1}(x_j + x_k)^{-1} \\ &= -\tfrac{1}{2}x_j^2 + 2 \sum_{k=1}^{\infty}{}' (x_j^2 - x_k^2)^{-1}, \end{aligned} \tag{5.10}$$

where on the left side of (5.10), $2\nu \neq -1, -2, -3,...$, and on the right side $\nu \neq -1, -2, -3,....$ Using (5.9) it follows that

$$2 \sum_{k=1}^{\infty}{}' (x_j^2 - x_k^2)^{-1} = -(\nu + 1)\, x_j^{-2}, \qquad \nu \neq -1, -2, -3,..., \tag{5.11}$$

which is the same as (5.3a) with $x_j^2 = y_j$. Note moreover that the reciprocal power sum (2.7) for the zeros x_j of the Bessel functions becomes

$$\sum_{j=1}^{\infty} x_j^{-2} = [2(\nu + 1)]^{-1}, \tag{5.12}$$

which of course reduces to (5.4) by considering the symmetry property of the zeros, i.e., multiplying the right-hand side by a factor of $\frac{1}{2}$. Note incidentally that the reciprocal power sums for the odd powers of the zeros x_j vanish

$$\sum_{j=1}^{\infty} x_j^{-3} = \sum_{j=1}^{\infty} x_j^{-5} = 0, \tag{5.13}$$

which follows from (2.8) and (2.10), or in general

$$\sum_{j=1}^{\infty} x_j^{-2m-1} = 0, \qquad m = 1, 2,.... \tag{5.14}$$

6. Zeros of Laguerre Polynomials

If a is zero or a negative integer, the confluent hypergeometric function $\Phi(a, c; x)$ becomes a polynomial in x, related to the generalized Laguerre polynomial $L_n^{\alpha}(x)$ by the formula [1, 13]

$$\Phi(-n, \alpha + 1; x) = \frac{n!}{(\alpha + 1)_n} L_n^{\alpha}(x). \tag{6.1}$$

Consequently replacing a and c by $-n$ and $(\alpha + 1)$, respectively, in the results for the zeros of the confluent hypergeometric function and noting that all the sums are now finite, we obtain from Theorems 1 to 5 the algebraic equations for the zeros z_j of the generalized Laguerre polynomial $L_n^{\alpha}(x)$,

$$2 \sum_{k=1}^{n}{}' (z_j - z_k)^{-1} = 1 - (1 + \alpha)\, z_j^{-1}, \tag{6.2a}$$

$$12 z_j^2 \sum_{k=1}^{n}{}' (z_j - z_k)^{-2} = -z_j^2 + 2(2n + \alpha + 1)\, z_j - (\alpha + 1)(\alpha + 5), \tag{6.2b}$$

$$8z_j^3 \sum_{k=1}^{n}{}' (z_j - z_k)^{-3} = (2n + \alpha + 1)\, z_j - (\alpha + 1)(\alpha + 3), \tag{6.2c}$$

$$\begin{aligned} 720 z_j^4 & \sum_{k=1}^{n}{}' (z_j - z_k)^{-4} \\ &= z_j^4 - 4(2n + \alpha + 1)\, z_j^3 \\ &\quad + 2[8n(n + \alpha + 1) + (\alpha + 1)(3\alpha + 1)]\, z_j^2 \\ &\quad -4(2n + \alpha + 1)(\alpha^2 - 19)\, z_j + (\alpha + 1)(\alpha^3 - \alpha^2 - 109\alpha - 251), \end{aligned} \tag{6.2d}$$

$$\begin{aligned} 288 z_j^5 & \sum_{k=1}^{n}{}' (z_j - z_k)^{-5} \\ &= -(2n + \alpha + 1)\, z_j^3 + [8n(n + \alpha + 1) + (\alpha + 1)(3\alpha + 1)]\, z_j^2 \\ &\quad - 3n(2n + \alpha + 1)(\alpha^2 - 9)\, z_j + (\alpha + 1)(\alpha + 5)(\alpha^2 - 6\alpha - 19), \end{aligned} \tag{6.2e}$$

where $j = 1, 2, \ldots, n$ and $\alpha \neq -1, -2, -3, \ldots$. These equations were recently derived in the context of the study of equilibrium configurations of certain one-dimensional many-body problems [3–5]. However, these equations are more general being applicable both to the case of real and comple zeros. Other sums of the type [3]

$$\sum_{j=1}^{n} z_j^{-p}, \qquad \sum_{j=1}^{n} \sum_{k=1}^{n}{}' (z_j - z_k)^{-p}, \qquad p = 1, 2, \ldots$$

can be similarly derived from analogous results for the zeros of the confluent hypergeometric functions.

7. Concluding Remarks

The zeros of certain combinations of confluent hypergeometric function $\Phi(a, c; x)$ also satisfy the results for the zeros of $\Phi(a, c; x)$. This is inferred from the recurrence relations for $\Phi(a, c; x)$. For instance, the relation [13]

$$(a - c + 1)\, \Phi(a, c; x) = (c - 1)\, \Phi(a, c - 1; x) - a\Phi(a + 1, c; x) \tag{7.1}$$

implies that the zeros of the contiguous function $(c - 1)\, \Phi(a, c - 1; x) - a\Phi(a + 1, c; x)$ satisfy all the results concerning the zeros of $\Phi(a, c; x)$, except, of course, for the case $a = c - 1$. The differential property [13]

$$\frac{d}{dx} \Phi(a, c; x) = \frac{a}{c}\, \Phi(a + 1, c + 1; x) \tag{7.2}$$

implies that the zeros of the first derivative of $\Phi(a, c; x)$ satisfy similar results as for $\Phi(a, c; x)$ with the replacement $a \to a + 1$ and $c \to c + 1$.

Nonlinear equations satisfied by the zeros of confluent hypergeometric function of course include results for zeros of other functions which are special cases [1] of this function such as Hermite polynomials, Whittaker functions, Parabolic Cylinder Functions and Coulomb Wave Functions.

A further generalization so as to obtain nonlinear equations of the form

$$S_m = \sum_{\substack{k=1 \\ k \neq j}} (x_j - x_k)^{-m}, \qquad m = 1, 2, 3, \ldots \tag{7.3}$$

for the zeros of other special functions not included here is also possible [21].

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